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Functional Equations for Circle Homeomorphisms with Golden Ratio Rotation Number

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The investigation of a scaling limit for mappings of the circle to itself with golden ratio rotation number leads to a pair of functional equations with at least a formal resemblance to the functional equation using the accumulation of period-doubling bifurcations. We discuss the general theory of these functional equations, assuming that solutions exist.

KEY WORDS: Golden-ratio; rotation number; functional equation; renormalization group.

1. INTRODUCTION

The investigation of a "scaling limit" for iterates of mappings of the circle with rotation number equal to the golden ratio $(\sqrt{5}-1)/2$ and with a critical point leads to the functional equations

$$g(x) = \alpha g\big(g(\alpha^{-2}x)\big) \tag{A1}$$

$$g(x) = \alpha^2 g\big(\alpha^{-2} g(\alpha^{-1}x)\big) \tag{B}$$

(The analysis leading to these equations will be outlined in Section 2.) Here, α is a number and $g(x)$ a function defined on some interval; both α and $g(x)$ are to be determined. We are going to consider only solutions with (i) $\alpha < -1$, (ii) $g(x)$ a strictly decreasing function of x, and (iii) $g(0) = 1$. The condition $g(0) = 1$ is simply a normalization; if $\bar{g}(x)$ is a solution of

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either equation with $\bar{g}(0) \neq 0$, then

$$g(x) = (\bar{g}(0))^{-1} \bar{g}(\bar{g}(0) \cdot x)$$

is also a solution (with the same α) satisfying $g(0) = 1$. We stress that the condition that g is strictly decreasing means

$$g(x_1) < g(x_2) \qquad \text{whenever} \quad x_1 > x_2$$

it does not rule out the existence of isolated points where g has vanishing derivative.

The analysis leading to the equations (A1) and (B) suggests that there should exist a function $g(x)$ which satisfies *both* of these equations. Furthermore, either of these equations, by itself, can be solved numerically, and the solution found appears, within computational error, to satisfy the other equation "automatically." These considerations lead to the suspicion that, under appropriate conditions, (A1) and (B) might be equivalent. A major step in the direction of proving this was made by Nauenberg,[4] who showed that an analytic solution of (A1) which satisfies in addition

$$g(\alpha^{-2}) = \alpha^{-2} \tag{A2}$$

must also satisfy (B). We will prove in Section 4 that, under mild differentiability and domain conditions, a solution of (B) also satisfies (A1). Note that condition (A2) follows immediately from (B) by putting $x = 0$ and using $g(0) = 1$. Thus, schematically, (A1) and (A2) together are equivalent to (B). *From now on, we will write* (A) *to denote the combination of conditions* (A1) *and* (A2).

It may be that (A2) is actually a consequence of (A1). It is not difficult to deduce from (A1) that $g^2(\alpha^{-2}) = \alpha^{-2}$, i.e., that α^{-2} is either a fixed point for g or a periodic point of period 2. The only solutions to (A1) (and the subsidiary conditions above) found so far do satisfy (A2), but, so far as I know, no general proof has been given.

Numerical computation of a solution to (A1) or (B) produces a function defined on a finite interval (or a bounded domain in the complex plane). In the analogous case of the functional equation for period-doubling (see, e.g., Collet and Eckmann[1]), the functional equation itself gives immediately an extension of the solution to the whole real axis. We investigate in Section 3 the problem of extending solutions of (B) defined on a finite interval to a larger set. What we show is that any solution defined on an interval of finite but reasonable size can be extended (uniquely) either to a solution defined on the whole real axis or to one defined on a finite interval which goes to infinity at both ends of that interval. The extended solution is as regular as the one we started from.

We also show that any solution arising as a scaling limit of a circle mapping with rotation number $(\sqrt{5}-1)/2$ is extendable to the whole real axis.

Section 5 is also devoted to an extension argument. As Feigenbaum et al.[2] have remarked, equation (A)—in contrast to (B)—makes sense for functions defined only on the interval $[0, 1]$. We show that any solution of (A) on $[0, 1]$ can be extended to a function defined on a considerably larger interval and satisfying both (A) and (B). (The extension theorem for solutions of (B) can then be used to continue this solution still further.) The arguments used in this section are straightforward adaptations of ideas in Nauenberg[4]; the main interest of the argument we give is that it separates clearly the algebraic and analytic elements in Nauenberg's argument.

2. THE SCALING LIMIT FOR CIRCLE MAPPINGS

In this section we describe some fine structure in the behavior of certain mappings of the circle to itself. Almost nothing in this section is new; we will summarize, from a slightly different point of view, the results of Feigenbaum, Kadanoff, and Shenker.[2] An alternative, and more or less equivalent, development has been given by Ostlund, Rand, Sethna, and Siggia.[5]

Let $f(x)$ denote a continuous strictly increasing mapping of the real line $\mathbf{R}$ to itself satisfying

$$f(x+1) = f(x) + 1 \tag{2.1}$$

By passage to quotients, such an f induces a one–one continuous mapping of the circle—represented as $\mathbf{R}/\mathbf{Z}$—onto itself, and any such mapping which is orientation preserving (increasing) is induced by an f which is unique up to an additive integer constant.

If f is as above, then

$$\lim_{n\to\infty} \frac{f^n(x) - x}{n}$$

exists for all x and is independent of x. This limit is called the *rotation number* of f; we will denote it by $\rho(f)$.

The mappings of the circle we want to discuss will be ones induced by f's with rotation number $(\sqrt{5}-1)/2$, the *golden ratio*. For the remainder of this paper, the symbol σ will be reserved to denote $(\sqrt{5}-1)/2$. We need some preliminaries on the relation between σ and the *Fibonacci sequence*, i.e., the sequence (Q_n) of integers satisfying

$$Q_{n+1} = Q_n + Q_{n-1}, \qquad Q_0 = 0, \qquad Q_1 = 1 \tag{2.2}$$

A well-known but relatively crude relation is

$$\lim_{n\to\infty} \frac{Q_{n-1}}{Q_n} = \sigma$$

This relation is sharpened considerably by the identity

$$Q_n \cdot \sigma - Q_{n-1} = (-1)^{n-1}\sigma^n \tag{2.3}$$

which can easily be proved by solving the recursion relation defining the Fibonacci sequence to get the explicit formula

$$Q_n = \left(\sigma^{-n} + (-1)^{n+1}\sigma^n\right)/(\sigma^{-1} + \sigma)$$

We give the identity (2.3) a "dynamical" interpretation as follows: Let $f(x)$ have rotation number σ. It follows at once from the definition of rotation number that

$$f_{(n)}(x) \equiv f^{Q_n}(x) - Q_{n-1}$$

has rotation number

$$Q_n \cdot \sigma - Q_{n-1} = (-1)^{n-1}\sigma^n$$

For n large, the rotation number of $f_{(n)}$ is small, so we might expect

$$f_{(n)}(x) \to x \qquad \text{as} \quad n \to \infty$$

We are going to look in detail at the limiting behavior of the $f_{(n)}$'s for functions f which are smooth but which, although strictly increasing, have zero as a critical point:

$$f'(0) = 0$$

and hence whose inverses are not smooth. A typical example is

$$f(x) = x + \omega_0 - \frac{1}{2\pi}\sin(2\pi x)$$

with the constant ω_0 adjusted to make the rotation number equal to σ. We want to concentrate particularly on the behavior near the critical point at zero, and we therefore magnify as follows: Let

$$\alpha^{(n)} = f_{(n)}(0)^{-1}$$

and

$$f_n(x) = \alpha^{(n-1)} f_{(n)}(x/\alpha^{(n-1)})$$

(We have chosen to rescale by $\alpha^{(n-1)}$ rather than $\alpha^{(n)}$ because this leads to slightly simpler formulas later on.) In the situation we want to look at, the $\alpha^{(n-1)}$ will tend to infinity, so the large-n behavior of f_n's on a fixed interval will reflect the behavior of $f_{(n)}$ and hence f^{Q_n} very near to zero.

As we have already remarked, $f_{(n)}$ has rotation number $(-1)^{n-1}\sigma^n$. This implies that

$$f_{(n)}(x) > x \qquad \text{everywhere for } n \text{ odd}$$
$$f_{(n)}(x) < x \qquad \text{everywhere for } n \text{ even}$$

and in particular that $\alpha^{(n)}$ has sign $(-1)^{n-1}$. We also note for use later that

$$f_n(x) < x \qquad \text{for all } n, x \tag{2.4}$$

Numerical experiments suggest that the following phenomenology is common for mappings f of the sort we are considering:

(i) The sequence of ratios $\alpha^{(n+1)}/\alpha^{(n)}$ approaches a limit $\alpha < -1$ (so that, roughly, $\alpha^{(n)} \approx \alpha^n$).

(ii) The sequence of functions f_n approaches a limit f^*.

(iii) The limiting ratio α and function f^* are *universal*, i.e., do not seem to depend on what f we start with in the class we are considering.

Note that the condition $f(x+1) = f(x)$ imposed on f will (presumably) *not* be satisfied by f^*. The functions f_n satisfy

$$f_n(x + \alpha^{(n-1)}) = f_n(x) + \alpha^{(n-1)}$$

and, since the $\alpha^{(n-1)}$ go to infinity, the condition becomes empty in the limit.

Independent of questions of universality, some striking consequences follow just from the convergence of the sequence f_n derived from a single f with rotation number σ. Suppose we have such an f. Write

$$\alpha_n = \alpha^{(n)}/\alpha^{(n-1)}$$

Then, first of all,

$$f_n(0) = \alpha^{(n-1)}f_n(0) = \alpha^{(n-1)}/\alpha^{(n)} = \alpha_n^{-1}$$

and so

$$\alpha^{-1} = \lim_{n\to\infty} f_n(0) = f^*(0)$$

From

$$Q_{n+1} = Q_n + Q_{n-1}$$
$$f_{(n)} = f^{Q_n} - Q_{n-1}$$
$$f(x+1) = f(x) + 1$$

it follows easily that

$$f_{(n+1)} = f_{(n)}f_{(n-1)} = f_{(n-1)}f_{(n)}$$

Rescaling by $\alpha^{(n)}$ and reorganizing in a straightforward way we get

$$f_{n+1}(x) = \alpha_n f_n\big(\alpha_{n-1} f_{n-1}(\alpha_n^{-1}\alpha_{n-1}^{-1}x)\big) \tag{2.5A}$$

from the first equality and

$$f_{n+1}(x) = \alpha_n \cdot \alpha_{n-1} f_{n-1}\big(\alpha_{n-1}^{-1} f_n(\alpha_n^{-1} x)\big) \tag{2.5B}$$

from the second. Taking limits as $n \to \infty$ we find

$$f^*(x) = \alpha f^*\big(\alpha f^*(\alpha^{-2} x)\big) \tag{2.6A}$$

$$f^*(x) = \alpha^2 f^*\big(\alpha^{-1} f^*(\alpha^{-2} x)\big) \tag{2.6B}$$

To summarize: *Using nothing but the assumption that appropriately magnified versions of $f^{Q_n}(x) - Q_{n-1}$ converge to a limit f^*, we have shown that f^* satisfies two functional equations.* Since solutions of these functional equations can be expected to be scarce, we get a simple intuitive explanation for "universality," i.e., the fact that many different f's produce the same f^*. Renormalization group ideas lead to an elaboration of this simple explanation for which we refer to Feigenbaum et al.,[2] MacKay,[3] or Ostlund et al.[5]

To get the functional equations (A1) and (B) as given in Section 1, we write

$$g(x) \equiv \alpha f^*(x)$$

and rewrite (2.6A) and (2.6B) accordingly. Since $\alpha^{-1} = f^*(0)$, the normalization condition $g(0) = 1$ is automatically satisfied. Since each f_n is increasing, f^* will be an increasing function so $g(x)$ will be a decreasing function. (Recall that α is negative and larger than one in magnitude.) We observed above that $f_n(x) < x$ for all n, x; hence,

$$f^*(x) \leqslant x \quad \text{and} \quad g(x) \geqslant \alpha x \tag{2.7}$$

at all points x where

$$f^*(x) = \lim_{n\to\infty} f_n(x)$$

The purpose of this paper is to investigate the properties of solutions of the equations (A1) and (B), assuming that the solutions exist. It may nevertheless be useful to sketch briefly the state of our knowledge about existence of analytic solutions. There is, first of all, a "trivial" solution

$$g(x) = 1 - x/\sigma, \qquad \alpha = -1/\sigma$$

This solution is what is obtained by taking the scaling limit described above for a smooth mapping with rotation number σ and derivative strictly positive everywhere. What we really want is a solution with $g'(0) = 0$. Monotonicity of g then implies $g''(0) = 0$, and it is natural to seek a solution with $g'''(0) \neq 0$. Assuming this, it is straightforward to show from either functional equation that $g^{(j)}(0) = 0$ for all j which are not multiples of 3, i.e., that g is an analytic function of x^3.

The usual way to solve (A1) or (B) numerically is to (i) rewrite the functional equation in terms of the variable x^3, (ii) truncate the functional equation to a finite number of conditions either by requiring only that it hold at a finite set of points or by requiring only that it hold to finite order at some point, (iii) use Newton's method to seek a polynomial satisfying the resulting finite set of nonlinear equations.

With a little care, Newton's method can be made to converge, and the result obtained always seems to be the same (i.e., independent of which equation is being solved and—within limits— of the truncation procedure used.)

These numerical results strongly suggest that there is function defined and analytic on a substantial interval of the real axis, satisfying (A) and (B), with

$$\alpha \approx -1.2885745\ldots$$

and with a second-order critical point at 0. It appears to be feasible to give a computer-assisted proof that this solution does in fact exist, and there are currently in progress at least two attempts to construct such a proof, one by D. Rand and B. Mestel, the other by R. de la Llave and the author.

3. EXTENSION OF SOLUTIONS OF (B)

We are going to prove in this section the existence of a maximal extension of a given solution of (B). We have first to specify carefully what we mean by a solution. We will, as always, only be concerned with *continuous, monotone-decreasing* solutions with $\alpha < -1$. Let $g(x)$ be a function defined on an interval I. For the sake of concreteness, we will write the formulas as if I is a closed interval $[a,b]$, but open and half-open intervals are also allowed. The domain I' of the function $\alpha^2 g(\alpha^{-2} g(\alpha^{-1}x))$, i.e., the set of numbers x such that

$$a \leqslant \alpha^{-1}x \leqslant b \quad \text{and} \quad a \leqslant \alpha^{-2}g(\alpha^{-1}x) \leqslant b$$

is either an interval or empty. For g to be a solution to (B), it is necessary at the very least that the following be true:

(1) $I \cap I'$ contains more than one point.

(2) $g(x)$ and $\alpha^2 g(\alpha^{-2} g(\alpha^{-1}x))$ agree on $I \cap I'$.

Condition (1) implies that 0 is in I, so the normalization condition $g(0) = 1$ makes sense. For our purposes, we add the following to these minimal conditions:

(3) The interval of definition $I = [a,b]$ is not too asymmetric:

$$|\alpha|^{-1} \leqslant \frac{|b|}{|a|} \leqslant |\alpha|$$

(4) $g(b) < 0$.

We will explain below our reasons for imposing these conditions. First, however, we note the following consequence:

Proposition 3.1. With the above assumptions, the equation

$$g(x) = \alpha^2 g\big(\alpha^{-2} g(\alpha^{-1} x)\big) \tag{B}$$

holds for all $x \in [a, b]$.

In other words, $[a, b]$ is a *self-defining interval* for (B).

Proof. We define

$$q(x) \equiv \alpha^{-2} g(\alpha^{-1} x)$$

so (B) reads

$$g(x) = \alpha^2 g(q(x))$$

Note that $q(x)$ is *increasing*. Also, let $I \cap I' = [a'', b'']$. What we want to show is that $a'' = a$, $b'' = b$.

From (3), $q(x)$ is defined on all of I, and from this it follows that either $b'' = b$ or $q(b'') = b$. If $b'' < b$, then

$$g(b'') = \alpha^2 g(q(b'')) = \alpha^2 g(b)$$

But $g(b) < 0$ by (4), and $\alpha^2 > 1$, so it follows that $g(b'') < g(b)$. Since $g(x)$ is decreasing and $b'' < b$, this is impossible, and so $b'' = b$. It can be shown in a similar way that $a'' = a$ [using, this time, the fact that $g(a) \geqslant g(0) = 1 > 0$]. ■

We can now explain why we impose (3). Suppose, for example, that $g(x)$ is defined on $[a, b]$ with $|a| > |\alpha|\,|b|$, (i.e., with $a < \alpha b$) and satisfies (B) on the interval $[a'', b'']$ where both sides are defined. The argument just given can readily be adapted to show that $a'' = \alpha b$ and that $q(a'') > a''$. Thus, the values taken on by $q(x)$ for $a'' \leqslant x \leqslant b''$ are all to the right of αb. Hence the values of $g(x)$ for $a \leqslant x < \alpha b$ do not in any way enter into Eq. (B), so $g(x)$ can be changed arbitrarily on that interval and remain a solution of (B). Condition (3) avoids this arbitrariness. Note that, in the case considered, we could simply replace a by αb; the restricted g would then satisfy (3).

We impose (4) simply because the functional equation has the form

$$g(x) = \alpha^2 g(q(x))$$

and thus offers no straightforward way to determine any negative values of g starting from positive values only.

Proposition 3.2. A solution of (B) satisfying conditions (1)–(4) above can be extended uniquely to a function defined and satisfying (B) everywhere on an interval (a_∞, b_∞) with either

$$a_\infty = -\infty, \qquad b_\infty = \infty$$

or

$$\lim_{x \downarrow a_\infty} g(x) = +\infty, \qquad \lim_{x \uparrow b_\infty} g(x) = -\infty$$

Proof. From $g(0) = 1$, $g(b) < 0$, it follows that there is an x_0 in $[0, b]$ with

$$g(x_0) = 0$$

From

$$g(x) = \alpha^2 g(q(x))$$

and the fact that g is decreasing, it follows that

$$\begin{aligned} q(x) &< x \qquad \text{if} \quad x > x_0 \\ q(x) &> x \qquad \text{if} \quad x < x_0 \end{aligned} \tag{3.1}$$

We will use these inequalities repeatedly.

Consider now the functional equation written in the form

$$g(x) = \alpha^2 g(q(x)) \tag{3.2}$$

We will use the right-hand side of this equation to extend g and so want to show that its domain of definition, which we denote by $[a_1, b_1]$, properly contains the original domain $[a, b]$ of g. This is immediate: Since $q(x)$ is defined on the interval $[\alpha b, \alpha a]$ which properly contains $[a, b]$, and, by (3.1),

$$q(a) > a, \qquad q(b) < b$$

We thus use the functional equation to extend g to $[a_1, b_1]$; the extended g is still strictly decreasing and is as regular as the unextended g. The functional equation holds, by construction, everywhere on $[a_1, b_1]$, and it is easy to check that condition (3) is preserved, i.e., that

$$|\alpha|^{-1} \leqslant \frac{|a_1|}{|b_1|} \leqslant |\alpha|$$

Note that (i) if $b_1 \neq \alpha a$ then $q(b_1) = b$ and (ii) if $a_1 \neq \alpha b$ then $q(a_1) = a$.

The extension process can be repeated to generate a solution of (B) defined on an increasing sequence of intervals

$$[a, b] \subset [a_1, b_1] \subset [a_2, b_2] \subset \cdots$$

and hence defined on (a_∞, b_∞) where

$$a_\infty = \lim_{n \to \infty} a_n, \qquad b_\infty = \lim_{n \to \infty} b_n$$

It is clear from the construction that the extension is unique.

Since

$$|\alpha|^{-1} \leqslant \frac{|a_n|}{|b_n|} \leqslant |\alpha| \tag{3.3}$$

both a_∞ and b_∞ are finite if one of them is. To complete the proof of the proposition, we need only show that, if a_∞ and b_∞ are finite, then

$$\lim_{x \downarrow a_\infty} g(x) = +\infty, \qquad \lim_{x \uparrow b_\infty} g(x) = -\infty$$

We will give the proof only for the first of these relations; the second is completely analogous.

From (3.3),

$$\alpha b_\infty \leqslant a_\infty$$

and we start by considering the case

$$\alpha b_\infty < a_\infty$$

Then, for sufficiently large n,

$$\alpha b_{n-1} < a_\infty \leqslant a_n$$

But, as we noted above (in the case $n = 1$), if $a_n \neq \alpha b_{n-1}$, then $q(a_n) = a_{n-1}$. Combining this remark with the functional equation, we get that

$$g(a_n) = \alpha^2 g(q(a_n)) = \alpha^2 g(a_{n-1})$$

for all sufficiently large n. Since $g(a_n) > 0$ for all n, it follows that

$$\lim_{n \to \infty} g(a_n) = \lim_{x \downarrow a_\infty} g(x) = \infty$$

We now eliminate the hypothesis

$$\alpha b_\infty < a_\infty$$

by showing that the alternative

$$\alpha b_\infty = a_\infty$$

leads to a contradiction. Thus, suppose this latter equality holds. Then

$$\alpha a_\infty > b_\infty$$

and hence, by an argument analogous to the one just given,

$$\lim_{x \uparrow b_\infty} g(x) = -\infty$$

But, as $x \downarrow a_\infty$, $\alpha^{-1}x \uparrow b_\infty$, so $q(x) = \alpha^2 g(\alpha^{-1}x) \downarrow -\infty$. This, however, eventually contradicts the fact (3.1) that $q(x) > x$ for $x < x_0$, and thus completes the proof. ■

To close this section, we show that, for a solution $g(x)$ obtained as outlined in Section 2 as a scaling limit of a circle map with golden ratio rotation number, a_∞ and b_∞ are necessarily infinite.

Proposition 3.3. Let the hypotheses be as in Proposition 3.2, but suppose in addition that there is an increasing function $f(x)$ with $f(x+1) = f(x)+1$ and with rotation number σ, such that (in the notation of Section 2) $f_n(x)$ converges to $\alpha^{-1}g(x)$ uniformly on compact sets in (a,b). Then $a_\infty = -\infty$, $b_\infty = +\infty$, and convergence is uniform on compact sets in $(-\infty,\infty)$.

Proof. If we can show that $f_n(x)$ converges to $\alpha^{-1}g(x)$ uniformly on compact sets in (a_∞, b_∞), it will follow from (2.7) that

$$g(x) \geqslant \alpha x$$

on (a_∞, b_∞). This bound, however, rules out the possibility that $g(x)$ goes to $-\infty$ for finite x; hence, it shows that b_∞ (and therefore also a_∞) must be infinite.

To prove uniform convergence on compact sets in (a_∞, b_∞), it suffices, by an obvious induction argument, to prove it on compact sets in (a_1, b_1).

Recall (2.5B):

$$f_{n+1}(x) = \alpha_n \cdot \alpha_{n-1} f_{n-1}\left(\alpha_{n-1}^{-1} f_n\left(\alpha_n^{-1}x\right)\right)$$

What we want to show is that the left-hand side of this identity converges uniformly on compact sets in (a_1, b_1) to $\alpha^{-1}g(x)$; we do this by proving convergence of the right-hand side to $\alpha g(q(x))$. Thus, let K be a compact set in (a_1, b_1). By the construction of the extension of g to (a_1, b_1), $\alpha^{-1}K$ and $q(K)$ are both contained in (a,b). Let K_1 be a compact set in (a,b) containing both $\alpha^{-1}K$ and $q(K)$ in its interior. For sufficiently large n, $\alpha_n^{-1}K$ is contained in K_1, so, from the uniform convergence of $f_n(x)$ to $\alpha^{-1}g(x)$ on K_1, it follows that $q_n(x) \equiv \alpha_{n-1}^{-1} f_n(\alpha_n^{-1}x)$ converges uniformly on K to $q(x)$. Similarly, $q_n(K) \subset K_1$ for all sufficiently large n, so uniform convergence of the right-hand side of (2.5) on K follows from the uniform convergence of f_{n-1} to $\alpha^{-1}g$ on K_1. ■

4. (B) IMPLIES (A)

In this section, $g(x)$ will denote a solution of (B) in the sense of Section 3, i.e., a strictly decreasing function with $g(0) = 1$, defined on an interval (a,b) and satisfying conditions (1)–(4) of Section 3. Invoking Proposition 3.2, we assume the g is maximally extended. As in Section 3, we let x_0 denote the unique point is (a,b) where $g(x_0) = 0$.

Proposition 4.1. If g is differentiable, if $g'(x_0) \neq 0$, and if the domain (a,b) of g contains 1, then

$$g(x) = \alpha g\left(g(\alpha^{-2}x)\right) \tag{A1}$$

everywhere on (a,b).

Remarks. 1. The assumption that $g'(x_0) \neq 0$ can be replaced by the assumption that, for some positive integer q, g is q times differentiable and $g^{(q)}(x_0) \neq 0$. (This extension seems, however, to be without practical interest.)

2. We actually need differentiability only at the two points x_0 and $\alpha^{-2}x_0$.

3. It is not very satisfactory to have to assume explicitly that g is defined at 1. (Note, however, that, if g is not defined at 1, the right-hand side of (A1) is not defined at 0.) This assumption can be replaced by the weaker one that the right and left sides of (A1) have at least one point of definition in common, but it has not, so far, been possible to eliminate it completely.

Proof. As in Section 3, we write

$$q(x) \equiv \alpha^{-2}g(\alpha^{-1}x)$$

We also write

$$r(x) \equiv g(\alpha^{-2}x)$$

The functional equations (A1) and (B) can now be rewritten, respectively, as

$$g(r(x)) = \alpha g(x) \tag{A1$'$}$$

$$g(q(x)) = \alpha^2 g(x) \tag{B$'$}$$

The obvious way for these equations to be consistent is to have

$$q(x) = r(r(x))$$

This identity is in fact a straightforward consequence of (B):

$$r(r(x)) = g\big(\alpha^{-2}g(\alpha^{-2}x)\big) = g\big(\alpha^{-2}g\big(\alpha^{-1}(\alpha^{-1}x)\big)\big)$$
$$= \alpha^{-2}g(\alpha^{-1}x) = q(x)$$

Note that, in this derivation, we have used (B) only at the point $\alpha^{-1}x$, so the identity holds for all x such that $\alpha^{-1}x \in (a,b)$, i.e., for all x in the domain of q.

We next argue that

$$r(x_0) = x_0$$

To see this, we note that r is a decreasing function on $[0,1]$ with $r(0) = 1$ and so has exactly one fixed point in $[0,1]$. This point is also fixed for $q = r^2$, and the only fixed point for q in (a,b) is x_0. [This argument is the only place where we need the assumption that g is defined at 1.]

Next: We differentiate (B) at x_0 and use $q(x_0) = x_0$ to get

$$g'(x_0) = \alpha^2 g'(x_0)q'(x_0)$$

and hence, since we are assuming $g'(x_0) \neq 0$,

$$q'(x_0) = \alpha^{-2}$$

Since $q = r^2$ and $r(x_0) = x_0$,

$$(r'(x_0))^2 = q'(x_0) = \alpha^{-2}$$

so, since r is decreasing and α negative

$$r'(x_0) = \alpha^{-1}$$

For any $x \in (a,b)$,

$$g(x) = \alpha^2 g(q(x)) = \cdots = \alpha^{2m} g(q^m(x))$$

Hence, as m goes to infinity, $g(q^m(x))$ goes to zero, so, since g is strictly decreasing,

$$q^m(x) \to x_0$$

We are first going to prove (A) assuming both $x \in (a,b)$ and $r(x) \in (a,b)$; then show that the second assumption is automatic. With the two assumptions, and using $q = r^2$, we get

$$\alpha g(r(x)) = \alpha^{2m+1} g(q^m(r(x))) = \alpha^{2m+1} g(r(q^m(x)))$$

We are going to show that

$$\lim_{m \to \infty} \alpha^{2m+1} g(r(q^m(x))) = g(x)$$

and hence that

$$\alpha g(r(x)) = g(x)$$

as desired.

Write x_m for $q^m(x)$. Then using $g(x_0) = 0$,

$$\alpha^{2m+1} g(r(x_m)) = \alpha \frac{g(r(x_m)) - g(x_0)}{r(x_m) - x_0} \cdot \frac{r(x_m) - x_0}{x_m - x_0} \cdot \alpha^{2m}(x_m - x_0) \quad (4.1)$$

On the other hand, by repeated application of (B),

$$g(x) = \alpha^{2m} g(x_m) = \frac{g(x_m) - g(x_0)}{x_m - x_0} \cdot \alpha^{2m}(x_m - x_0)$$

and so

$$\lim_{m \to \infty} \alpha^{2m}(x_m - x_0) = g(x)/g'(x_0)$$

Combining this with (4.1) and using $r'(x_0) = \alpha^{-1}$, we get

$$\lim_{m \to \infty} \alpha^{2m+1} g(r(x_m)) = \alpha g'(x_0) r'(x_0) g(x)/g'(x_0) = g(x)$$

This proves (A1) provided that both x and $r(x)$ are in the interval (a,b) on which g is defined. We now show that r maps this interval into itself so the second condition is automatic. Suppose not. Then a and b must be finite and either a or b must be the image under r of some point in (a, b). For definiteness, assume that there is an $x_a \in (a, b)$ with $r(x_a) = a$. Since r is decreasing, $x_0 < x_a < b$. By what we have just shown,

$$g(x) = \alpha g(r(x)) \qquad \text{for} \quad x_0 < x < x_a$$

Let x approach x_a from below. Then $r(x)$ approaches a from above, so $g(r(x))$ approaches ∞, so $g(x) = \alpha g(r(x))$ approaches $-\infty$. But this contradicts the assumption that $x_a < b$. ■

5. (A) IMPLIES (B)

As noted in Section 1, Nauenberg[4] has shown that an analytic solution of

$$g(x) = \alpha g\big(g(\alpha^{-2}x)\big), \qquad g(\alpha^{-2}) = \alpha^{-2} \tag{A}$$

also satisfies (B). (His argument can actually be made to work assuming much less than analyticity.) We will prove here a complementary result, purely algebraic, showing that a solution of (A) on [0, 1] can be extended to a larger interval where it satisfies both (A) and (B).

For this section, our assumptions will be that g is a strictly decreasing function defined on [0, 1], with $g(0) = 1$, satisfying (A) on [0, 1]. As always, we assume $\alpha < -1$.

It follows from (A1) with $x = 0$ that

$$\alpha^{-1} = g(1)$$

hence, that $g(1) < 0$; hence, that g vanishes somewhere on [0, 1]. As previously, we denote this point by x_0. Note that the right-hand side of (A1) is defined for x up to $\alpha^2 x_0$, so we can take it as defining an extension of g to $[0, \alpha^2 x_0]$, and the extended function will still satisfy (A). We will from now on assume that this extension has been made.

At this point, it makes very little sense to ask whether g satisfies

$$g(x) = \alpha^2 g\big(\alpha^{-2} g(\alpha^{-1}x)\big) \tag{B}$$

the left-hand side is defined only for positive x and the right-hand side only for negative x. The only place where the two sides can be compared is at 0, and there (B) reduces to $g(\alpha^{-2}) = \alpha^{-2}$, i.e., to (A2). The left-hand side of (B) can, however, be taken as *defining* $g(x)$ for $\alpha x_0 \leqslant x < 0$. (B) then holds, by definition, on that interval. It is also possible to ask whether (i) (A1) holds for $\alpha x_0 \leqslant x < 0$ or (ii) (B) holds for $0 < x \leqslant \alpha^2 x_0$.

We are going to show, by straightforward verifications, that the answers to both questions are affirmative. Thus: Starting with a solution of

(A) defined only on [0, 1], we can extend, using the functional equations, to a function defined on $[\alpha x_0, \alpha^2 x_0]$ and satisfying both (A) and (B).

Let $\alpha x_0 \leqslant x < 0$; we want to show that

$$g\big(g(\alpha^{-2}x)\big) = \alpha^{-1}g(x)$$

The calculation is as follows:

$$\begin{aligned}
g\big(g(\alpha^{-2}x)\big) &= g\big(\alpha^2 g\big(\alpha^{-2}g(\alpha^{-3}x)\big)\big) && (1)\\
&= \alpha g\Big(g\big(\alpha^{-2}\alpha^2 g\big(\alpha^{-2}g(\alpha^{-3}x)\big)\big)\Big) && (2)\\
&= \alpha g\Big(g\big(g\big(\alpha^{-2}g(\alpha^{-3}x)\big)\big)\Big) &&\\
&= \alpha g\Big(\alpha^{-1}\Big(\alpha g\Big(g\big(\alpha^{-2}g(\alpha^{-3}x)\big)\Big)\Big)\Big) &&\\
&= \alpha g\Big(\alpha^{-1}\Big(g\big(g(\alpha^{-3}x)\big)\Big)\Big) && (3)\\
&= \alpha g\Big(\alpha^{-2}\Big(\alpha g\Big(g\big(\alpha^{-2}(\alpha^{-1}x)\big)\Big)\Big)\Big) &&\\
&= \alpha g\big(\alpha^{-2}g(\alpha^{-1}x)\big) && (4)\\
&= \alpha^{-1}g(x) && (5)
\end{aligned}$$

Only the numbered steps are substantive; we now have to justify each of them. Steps (1) and (5) are just application of the definition of the extended g for negative values of its argument; all that has to be checked is that the argument is in $[\alpha x_0, 0)$, and this is immediate in each case. The other three steps each use

$$\alpha g\big(g(\alpha^{-2}z)\big) = g(z) \qquad \text{(A1)}$$

and it has to be checked in each case that the corresponding z is in $[0, \alpha^2 x_0]$. For step (4), $z = \alpha^{-1}x$, and it is immediate that this is in the appropriate interval. For step (3), $z = g(\alpha^{-3}x)$. Since $\alpha x_0 \leqslant x < 0$,

$$0 < \alpha^{-3}x \leqslant \alpha^{-2}x_0 < x_0$$

so

$$g(\alpha^{-3}x) \in \big[\,g(\alpha^{-2}x_0), g(0)\big] \subset [0,1] \subset [0, \alpha^2 x_0]$$

[To see that $1 < \alpha^2 x_0$, recall that

$$g(\alpha^{-2}) = \alpha^{-2} > 0 = g(x_0)$$

and thus $\alpha^{-2} < x_0$.]

For step (2),

$$z = g\big(\alpha^{-2}g(\alpha^{-3}x)\big)$$

From

$$g\big(g(\alpha^{-2}x_0)\big) = \alpha^{-1}g(x_0) = 0$$

it follows that

$$g(\alpha^{-2}x_0) = x_0$$

We have already verified that

$$g(\alpha^{-3}x) \in \big[\, g(\alpha^{-2}x_0), 1\big] = [x_0, 1]$$

Hence

$$g\big(\alpha^{-2}\big(g(\alpha^{-3}x)\big)\big) \in \big[\, g(\alpha^{-2}), g(\alpha^{-2}x_0)\big] = [\alpha^{-2}, x_0]$$

as desired.

The preceding calculation was used by Nauenberg in his proof that an analytic solution of (A) also satisfies (B).

We next want to verify that

$$g\big(\alpha^{-2}g(\alpha^{-1}x)\big) = \alpha^{-2}g(x) \qquad \text{for} \quad 0 \leqslant x \leqslant \alpha^2 x_0$$

The calculation is as follows:

$$\begin{aligned} g\big(\alpha^{-2}g(\alpha^{-1}x)\big) &= g\Big(\alpha^{-2}\big(\alpha^2 g\big(\alpha^{-2}g(\alpha^{-2}x)\big)\big)\Big) && (1)\\ &= \alpha^{-1}\alpha g\Big(g\big(\alpha^{-2}\big(g(\alpha^{-1}x)\big)\big)\Big) \\ &= \alpha^{-1}g\big(g(\alpha^{-2}x)\big) && (2)\\ &= \alpha^{-1}\alpha g\big(g(\alpha^{-2}x)\big) \\ &= \alpha^{-2}g(x) && (3) \end{aligned}$$

Step (1) is just the definition of $g(\alpha^{-1}x)$; step (3) is just (A1) applied at x. Step (2) is (A1) applied at $g(\alpha^{-2}x)$; we have to verify that this argument is in $[0, \alpha^2 x_0]$. But $0 \leqslant x \leqslant \alpha^2 x_0$ so $0 \leqslant \alpha^{-2}x \leqslant x_0$ so $g(\alpha^{-2}x) \in [g(x_0), g(0)] = [0, 1]$, as desired.

One thing the preceding analysis does *not* show is that, if g is analytic on a complex neighborhood of $[0, 1]$ (and hence defined for small negative x's), then our extension coincides with its analytic continuation. For this, we need the analytic part of Nauenberg's argument, which can be summarized as follows: Define

$$\tilde{g}(x) \equiv \alpha g\big(\alpha^{-2}g(\alpha^{-1}x)\big)$$

wherever the right-hand side is defined (and in particular on $[\alpha x_0, 0]$, where we used this function to extend g). The argument given above shows that

$$\tilde{g}(x) = \alpha g\big(\alpha^{-2}\tilde{g}(\alpha^{-1}x)\big) \qquad (5.1)$$

for small negative x and hence (by analytic continuation) on a complex neighborhood of zero. The problem is to show that $\tilde{g}$ and g agree near zero. This is done by showing that their Taylor series at zero coincide. There are two main steps:

1. Suppose $g(x) = 1 - ax^p + O(x^{p+1})$ with $a \neq 0$. Then $\tilde{g}(x)$ has the same form, with the same constant a. This follows from the definition of $\tilde{g}$ together with the formula

$$g'(\alpha^{-2}) = \alpha^p$$

which is easily deduced from (A).

2. For $j > p$, by differentiating (A1) and (5.1) repeatedly, putting $x = 0$, and rearranging, one gets formulas for $g^{(j)}(0)$ and $\tilde{g}^{(j)}(0)$ in terms of lower-order derivatives and derivatives of g at 1. From these expressions, it follows by induction on j that $\tilde{g}^{(j)}(0) = g^{(j)}(0)$ for $j = p+1, p+2, \ldots$. It also follows, incidentally, that $g^{(j)}(0) = 0$ unless j is a multiple of p, i.e., that g is an analytic function of x^p.

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